On the Characterization of the Scaling Behavior of Dissipative Dynamical Systems through a Generalized Entropy

R. Stoop, J. Parisi, and H. Brauchli

Physics Institute, University of Zurich, Zurich, Switzerland
and Institute for Mechanics, Federal Polytechnical University (ETH) of Zurich, Zurich, Switzerland

Z. Naturforsch. **46 a**, 642–644 (1991); received April 24, 1991

Different aspects of the scaling behavior of a dissipative dynamical system are obtained in a straightforward way from the associated generalized entropy function which, in this way, embodies the relevant information on the scaling behavior of the system. This is demonstrated by the application of the thermodynamic formalism to two model systems for each of which, however, a specific numerical approach must be followed in order to overcome the numerical problems.

Key words: Dissipative chaotic systems, Scaling behavior, Thermodynamic formalism.

1. Introduction

The usefulness of the generalized entropy function has recently been pointed out in a number of contributions [1–4]. Explicit examples, however, have been scarce. In this communication we give a detailed description of the results obtained from applying this formalism to two closely related model systems. In this way, some insight can be gained in the way how the generalized entropy function is able to store the information about the scaling behavior of a dissipative chaotic system.

As appropriate models for a hyperbolic dissipative dynamical system we consider three-scale Cantor sets (for a reference to the physical relevance of three-scale Cantor sets see, e.g., [5]). From the point of view of symbolic dynamics, such a system can be described by three symbols A, B, and C, to each of which a typical length and a probability is associated. For the simpler model we assume a complete grammar (a rather specific, nongeneric situation), whereas for the second, more realistic case it is assumed that the folding is incomplete or, to use different words, the symbolic tree is pruned. To have a specific example, let us require that the substring, $\ldots CCC \ldots$, cannot appear among the substrings of larger lengths $n > 3$. We, furthermore, assume that a hierarchical discovery has been made in the following sense: The probability which would have been attributed to a forbidden substring is distributed proportionally to its next branches on the symbol tree, while all other probabilities remain unchanged. Let us point out that, notwithstanding the fact that hyperbolic models are nongeneric, they are nonspecific with respect to the purpose for which they are considered here.

Reprint requests to Dr. R. Stoop, c/o Priv.-Doz. Dr. J. Parisi, Physics Institute, University of Zurich, Schönberggasse 9, CH-8001 Zurich, Switzerland.

2. The Generalized Entropy Function

For the thermodynamic formalism it is supposed that a generating partition has been found. Using such a partition consisting of M symbols, one proceeds in analogy to statistical mechanics: For an associated attractor or repeller A, the partition function Z_G is introduced [4]:

$$Z_G(q, \beta, n) = \sum_{j \in (1, \ldots, M)^n} l_j^{\beta} \, p_j^{q} \,. \tag{1}$$

Here the size of the j-th region R_j of the partition is denoted by l_j, whereas the probability of falling into this region is denoted by p_j ($p_j = \int_{R_j} \varrho(x)\, \mathrm{d}x$, where $\varrho(x)$ denotes the natural measure). To account for the nonisotropy of the attractor, they should be thought of as vectors. β and q are sometimes called "filtering exponents". Local scaling of l and p in n (where n denotes the "level" of the partition) is expected. In this way, the length scale l and the probability p give rise to scaling exponents ε and α through $l_j = e^{-n\varepsilon_j}$ and $p_j = l_j^{\alpha_j}$, respectively. These exponents are to be considered as vectors

0932-0784 / 91 / 0700-0642 $ 01.30/0. – Please order a reprint rather than making your own copy.

as well. Using the above expression, the generalized free energy F_G can be derived from the partition function [2]:

$$F_G(q, \beta) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{j \in (1, \ldots, M)^n} e^{-n\varepsilon_j(\alpha_j q + \beta)}, \quad (2)$$

where log denotes the natural logarithm. A generalized entropy function $S_G(\alpha, \varepsilon)$ is then introduced through the "global" scaling assumption that the number of regions N which have scaling exponents between (α, ε) and $(\alpha + d\alpha, \varepsilon + d\varepsilon)$ scales as $N(\alpha, \varepsilon)\, d\alpha\, d\varepsilon \sim e^{S_G(\alpha, \varepsilon)}\, d\alpha\, d\varepsilon$.

Writing the partition function formally as an integral, via a saddle-point approach the relationship between the generalized free energy F_G and the generalized entropy S_G is found as

$$S_G(\alpha, \varepsilon) = F_G(q, \beta) + (\langle\alpha\rangle q + \beta)\langle\varepsilon\rangle, \quad (3)$$

where the angular brackets indicate that those values of α and ε leading to the maximum of Z_G (as a function of given q and β) have been chosen. In the following, we will omit the brackets. The free energy F_G or the generalized entropy S_G describe in this way the scaling behavior of the dynamical system. Note that the information-theoretical "Renyi entropies" evolve from (2) for $\beta = 0$ [6].

From the generalized free energy and entropy, respectively, three more specific free energies and entropies can be derived by restriction: For $q = 0$, we obtain the free energy first discussed by Oono and Takahashi [7] as the maximum value of $S_G(\alpha, \varepsilon)$ with respect to the variation of α alone for given ε. The associated free energy and entropy are in this case denoted by $F_G(\beta)$ and $S_G(\varepsilon)$, respectively. For $q = 1$, one can derive the generalized Lyapunov exponents [8] and their entropy function (note the difference with the generalized Lyapunov exponents obtained from a sampling process). Furthermore, also the fractal dimensions [6, 9] can be obtained from $F_G(q, \beta)$ as the zeros $\beta_0(q)$ of $F_G(q, \beta)$ for given q [4]. The entropy-like function $f(\alpha)$ introduced in [7] (often called dimension spectrum) is then given by $S_G(\alpha_0, \varepsilon_0) = \varepsilon_0 f(\alpha_0)$, where ε_0 and α_0 lead to this zero of F_G for given q and appropriately chosen $\beta(q)$. From (3) it follows that $-\beta_0(q) = \alpha_0 q - f(\alpha_0)$. By using (1)–(3), the characteristic thermodynamic functions can, in principle, be calculated from an approximation of the system by strings of finite, but large enough lengths n.

3. Numerical Results

For the first model, the generalized entropy function can be calculated directly from the partition function. In Fig. 1, we display the support of this entropy function. The same approach for the more complicated model suffers from severe numerical difficulties. Here, a more recent approach via the zeta-function formalism put forward by Cvitanović [10] must be followed (physically, this can be interpreted as the use of the grandcanonical instead of the canonical ensemble). Results of this calculation are presented in Figure 2. It is easily seen that the support of the entropy function changes considerably, due to the fact that the symbolic dynamics of the system can most conveniently be described by redefining the alphabet. Figure 3 compares the entropy function for both model cases. A discussion of the convergence proper-

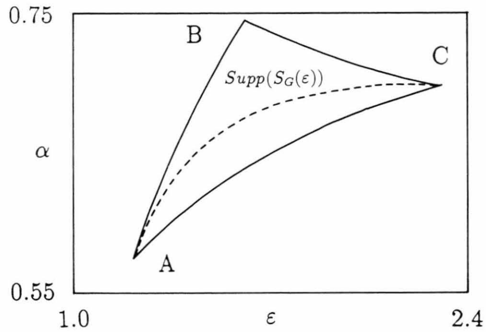

Fig. 1. Support of the entropy function $S_G(\alpha, \varepsilon)$ of the unrestricted three-scale Cantor set. For pairs (α, ε) within the closed area, $S_G(\alpha, \varepsilon)$ is a nonzero, convex function. The dashed line indicates the support of $S_G(\varepsilon) = S_G(\alpha, \varepsilon)|_{q=0}$. The corner points are determined by the scaling properties of the symbols A, B, and C, respectively.

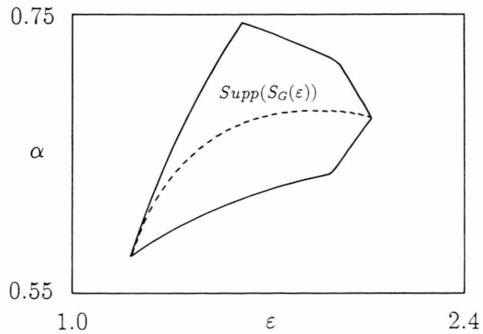

Fig. 2. Support of the entropy function $S_G(\alpha, \varepsilon)$ for the model with a nontrivial grammar, together with the support of $S_G(\varepsilon)$ (see text).

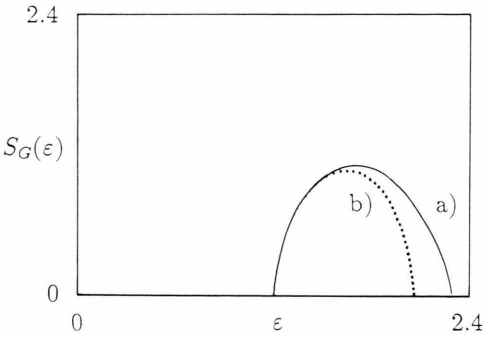

Fig. 3. Plots of the entropy function $S_G(\varepsilon)$ for the first (a) and the second model (b).

ties of the first approach applied to the second model, which is of interest itself but yields unsatisfactory results, will be given in a forthcoming publication [11].

To conclude, we believe that the material presented makes evident the usefulness of the description of the scaling behavior of a dissipative dynamical system via the generalized entropy function. It, furthermore, demonstrates what changes can be expected if a hierarchical discovery in the sense of the second model is being made. The application to nonhyperbolic maps is straightforward and needs not to be discussed here [11], whereas for experimental systems future applications are expected. Finally, we would like to point out the fact that in these most recent developments not only the definition of a chaotic attractor as the closure of its unstable periodic orbits finds its inherent justification, but also a way has been found to incorporate in one representative function all the relevant information of the scaling behavior of a dissipative dynamical system.

[1] R. Stoop and J. Parisi, Phys. Rev. A **43,** 1802 (1991).
[2] R. Stoop, J. Peinke, and J. Parisi, Physica D, to be published.
[3] T. Tél, Z. Naturforsch. **43a,** 1154 (1988).
[4] M. Kohmoto, Phys. Rev. A **37,** 1345 (1988).
[5] G. Gunaratne, M. H. Jensen, and I. Procaccia, Nonlinearity **1,** 157 (1988).
[6] A. Renyi, Probability Theory, North-Holland, Amsterdam 1970.
[7] Y. Oono and Y. Takahashi, Progr. Theor. Phys. **63,** 1804 (1980).
[8] H. Fujisaka, Progr. Theor. Phys. **70,** 1264 (1983).
[9] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. Shraiman, Phys. Rev. A **33,** 1141 (1986).
[10] P. Cvitanović, in: Nonlinear Physical Phenomena, Brasilia 1989 Winter School (A. Ferraz, F. Oliveira, and R. Osorio, eds.), World Scientific, Singapore 1990.
[11] R. Stoop and J. Parisi, to be published.

Verantwortlich für den Inhalt: A. KLEMM
Satz und Druck: Konrad Triltsch, Würzburg